Apollonius Conics Proposition I.58

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In Heath's version of Apollonius' Conics the preliminaries to proposition I.58 includes this: "…let the given angle not be a right angle but equal to the angle CPT, where C is the middle point of the given diameter PP'; and let PL be the parameter corresponding to PP'. Take a point N, on the semicircle which has CP for its diameter, such that NH drawn parallel to PT satisfies the relation

sq. NH : rect. CH,HP :: PL : PP'."

Heath indicates the proof proceeds without elaboration but also notes that Euoticus in the 5th or 6th century wrote a commentary on the Conics which includes a method for constructing NH. The clearest description of Euoticus' method is found in Colin McKinney's 2010 doctoral dissertation "*Conjugate diameters: Apollonius of Perga and Eutocius of Ascalon*." It is available from the University of Iowa at http://ir.uiowa.edu/etd/711/. The method presented here is a less complex method such as Apollonius could have used and thought no elaboration was needed.

Proposition I.58 Overview

In an earlier proposition Apollonius develops the relationship that exists between the tangents at the ends of any two diameters and the parameter for one of the diameters. If the parameter is known for one diameter the parameter can be found for any other diameter using that relationship.

I.58 is Apollonius method for finding the axis of an elliptical section when a diameter, the parameter for that diameter and the ordinate angle are known. He uses a bit of reverse engineering for this construction in that he knows what property an axis must have and proceeds to construct a diameter having those properties. Apollonius verifies that the diameter so constructed is an axis by using it to correctly produce the known parameter of the "given" diameter.

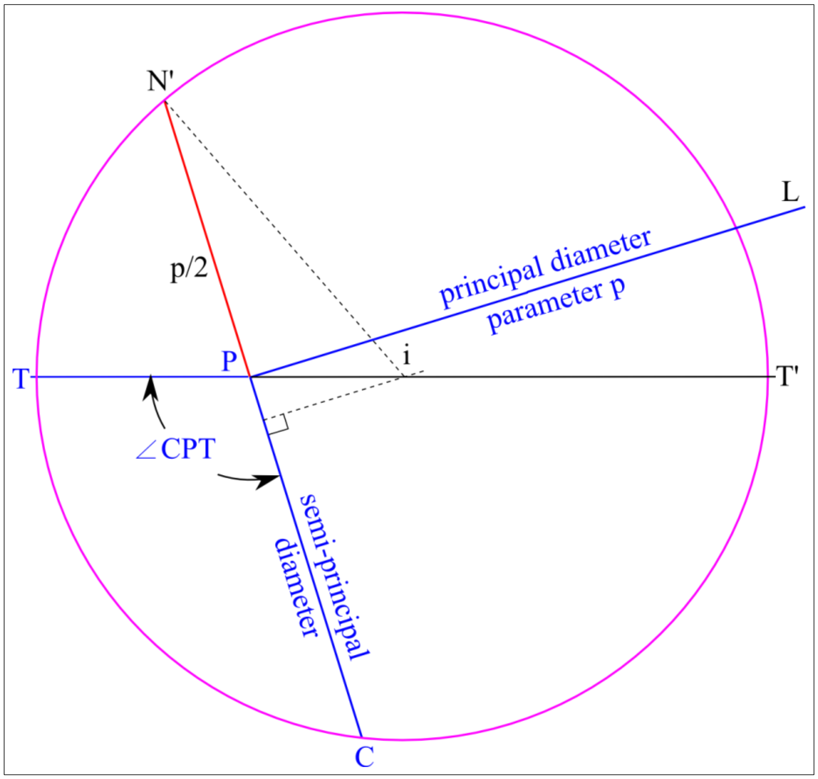
The length of the axis and the associated parameter are also found in the proposition but this tutorial stops after proving the constructed diameter is an axis. The remainder of the proposition is straight forward and the interested reader should refer to Apollonius' Conics.

Example Conic Section

The example section used in this tutorial has the following characteristics:

semi-principal diameter	CP = 55.67	half of PP'
semi-conjugate diameter	CD = 48.69	half of DD'
half parameter	p/2 = 42.58	half of PL
Ordinate Angle	OA = 72.89°	(acute angle of inclination; obtuse inclination will be the supplemental to this angle)
Major Axis Angle	MAA = 24.74°	measured relative to the principal diameter.

What's in a Circle.



There is saying that "a little knowledge is a dangerous thing" but when it comes to circles a bit of knowledge can make something difficult simple.

In the figure above, however, we start not with the circle, but by laying out the angle CPT, the principal diameter and the diameter's parameter. Heath indicates Apollonius' called ∠CPT the "angle of inclination between the diameter and its ordinates." This leaves an ambiguity as to whether it is the obtuse or the acute angle that is being referenced. They are supplemental and it usually works to match the angle appropriately. In the case of our example the given value of ∠OA indicates it to be an acute angle and thus I drew ∠CPT as an obtuse angle equal 180 – ∠OA degrees to match Heath's diagram.

With the givens in place, extend CP by half the length of the parameter to N'. The segments of the line CPN' will have the ratio: PN':CP = p/2:CP = p:2·CP = p:PP'.

Erect a perpendicular from the midpoint of CN' extend it to intersect TP extended at point i. With i as center draw a circle of radius iC = iN'. It will pass through C and N', making CN' a chord, and intersect Pi extended at T and T'. TT' is both a chord and a diameter.

From the property that products of intersecting chords are equal: CP·PN' = TP·PT'and it follows that:

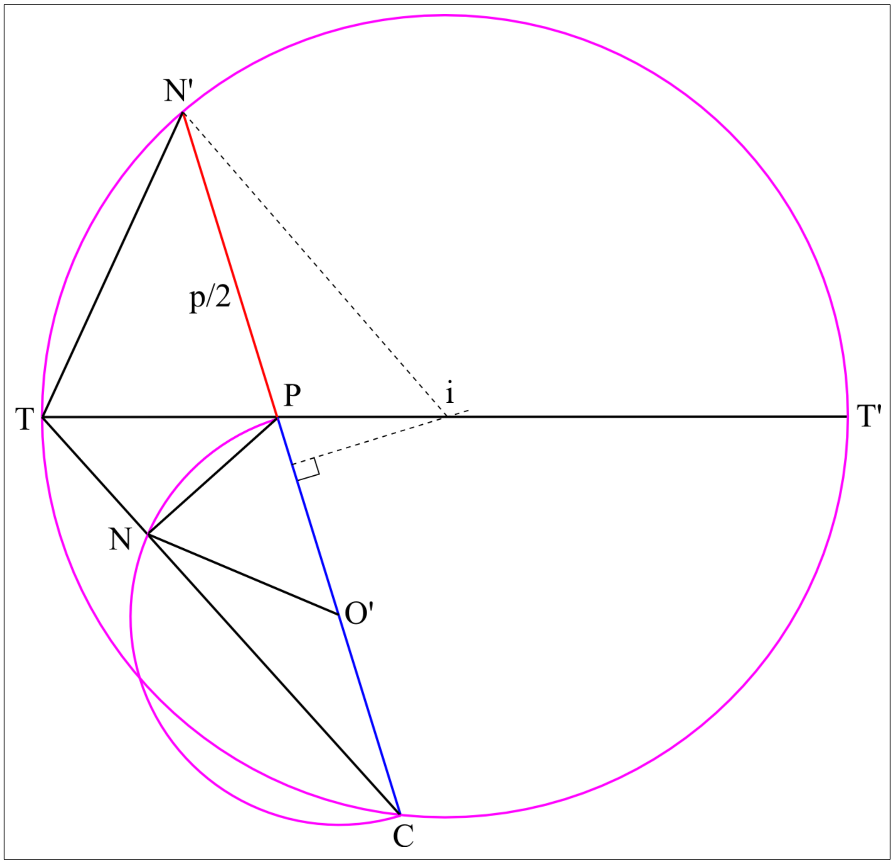
$$PN'^2:TP*PT' = PN'^2:CP*PN' = PN':CP = p:2*CP = p:PP'$$

showing that the ordinate PN' has the orientation and relationship to the segments of the diameter of a semi-circle that are required of the ordinate NH. The semi-circle TN'T' and ordinate PN' can indeed be duplicated, flipped, rotated, scaled, truncated and pasted onto CP in the original circle to produce NH.

Applying the Constructed to I.58

Rather than duplicating and scaling the previous construction we make use of it in a manner that simplifies the proof that our construction of NH is correct.

Draw CT. The measure of an arc is equal to its central angle and twice the angle of an angle



inscribed in the same circle. Arc TN' is intercepted by both the central angle ∠TiN' and the inscribed angle ∠TCN'. Thus angle ∠TCN' = ∠TiN'/2.

Draw a semi-circle on CP with center at O'. It will intersect the line CT at N. Note that point N could be located by dropping from P a perpendicular to the line CT. Drawing the semi-circle is an easy to way to construct the perpendicular since it will inscribe the angle ∠CNP and make it right. The semi-circle is also needed for the proof.

Arc NP is intercepted by both ∠NO'P and ∠NCP. Thus,

∠NO'P = 2·∠NCP = 2·∠TCN' = 2·(∠TiN'/2) = ∠TiN'.

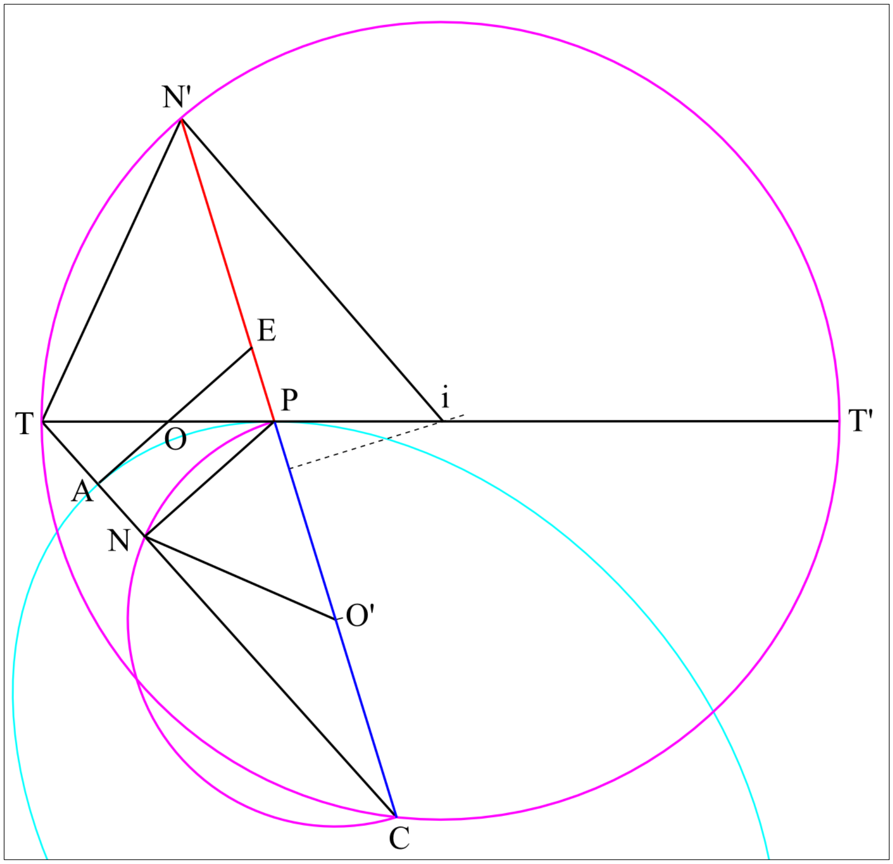
△TiN' and △NO'P are isosceles with equal apex angles which makes their base angles equal and the triangles similar.

Where's the curve? Curve construction is done in proposition I.56 or VI.30. Those propositions require that the major axis be known and I.58 is Apollonius' is method to find the axis from a known diameter. He does not assume the curve to be given.

I found the curve to be a help in understanding the construction of the proof and have added the curve from which the example data was taken to the next figure.

Proof that CA is an axis

Apollonius is constructing an axis and AE is to be the tangent to the curve at the end of the axis. The tangent at the ends of Axes is perpendicular to the axis. Accordingly AE must be



drawn perpendicular to CT at the point where CT and the curve of the section intersect. Remember you don't have either the curve or point A.

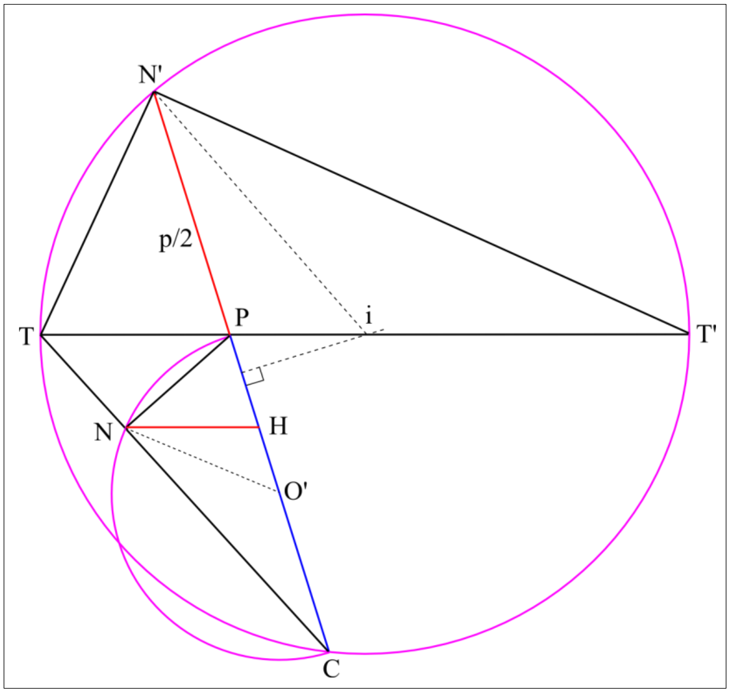
Proposition I.37 shows that the point N divides CT so that CA is the mean proportional between CN and CT. That is, CT:CA :: CA:CN. Find point CA and draw AE parallel to NP. A method for finding CA is given in the end notes.

AE and NP are parallel and cut by the line CN' hence ∠NPO' = ∠OEP.

It was shown above that △TiN' and △NO'P are isosceles and that their base angles are equal. Thus ∠PTN' = ∠NPO'. But ∠NPO' = ∠OEP because AE and NP are parallel lines cut by the same line. This in turn makes ∠PTN' = ∠OEP

∠OPE is common to both △OPE and △TPN' and ∠OEP = ∠PTN'. Hence △OPE and △TPN' are similar. The sides opposite equal angles in similar triangles are proportional and PN':PT = (p/2):PT = p:2·PT :: OP:PE. This is the same result that Apollonius arrived at to verify that CA is the required semi-axis.

Proof that the method used above correctly locates point N?

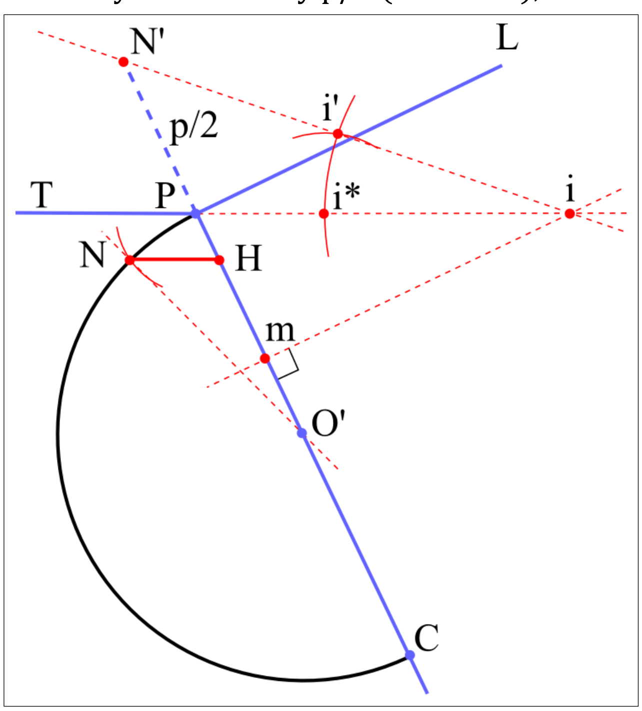


We did not use NH in the previous proof but it and the line N'T' have been added to the figure.

It was previously shown that
△TN'i ∽△PNO' and thus
∠NPC = ∠N'TT'. Both ∠TN'T' and ∠PNC are right and equal. Thus
△TN'T' ∽△PNC.
∠NPH = ∠N'TP and ∠NHP = ∠TPN' thus △NPH ∽△N'TP. △PN'T' and △HNC have equal angles making them similar. Hence
NH:PH :: N'P:TP and N'P:PT' :: NH:HC
NH^2:PH·HC :: $N'P^2$:TP·PT' and thus
NH^2:PH·HC = p:PP'

which completes the proof that our location of point N is correct.

Having proved the method correct, we can dispense with drawing the circle for finding N. We need only extend CP by p/2 (half of PL), locate i and duplicate the angle ∠PiN' at O'.

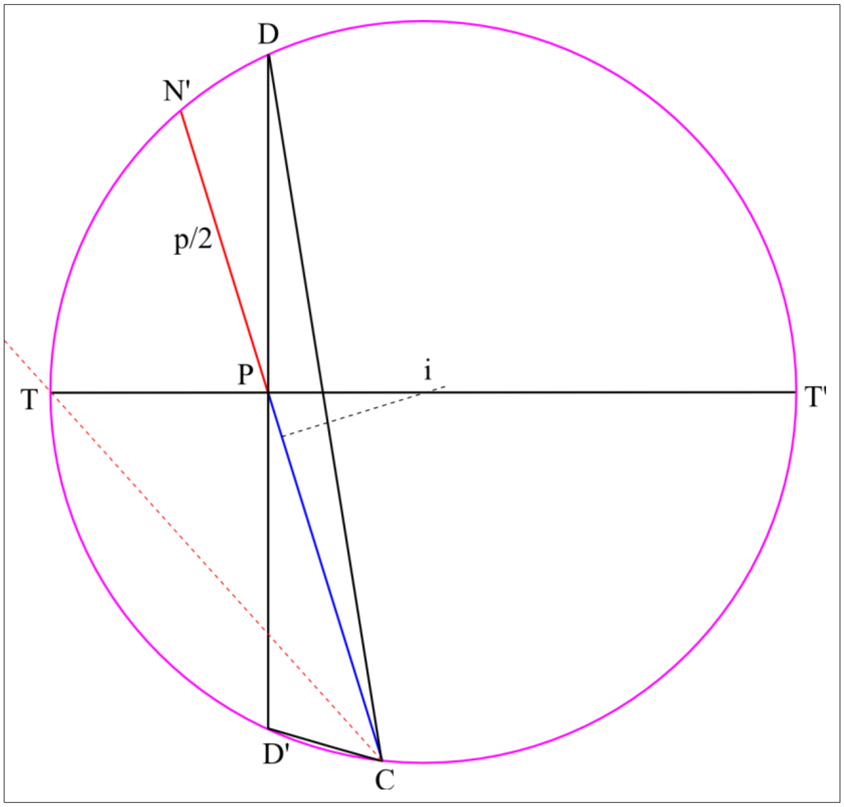


This simplified method is illustrated in the figure to the left using a section with different PP' and PL to make the drawing more compact.

It shows how the angle PiN' can be copied by first swinging an arc of radius O'P from i intersecting iP at i* and iN' at i'. The arc length i*i' is then marked from P intersecting the semi-circle at N so that ∠NO'P = ∠PiN'.

Finally, NH is drawn through N parallel to PT.

A Second Look at the Circle



A line drawn through the point P perpendicular to TT' will intersect the circle at D and D' as shown and is bisected by TT'. Hence, DP·D'P = DP^2 = PN'·CP , or, CP:DP :: DP:PN'.

This simply says that DP – the semi conjugate diameter - is the mean proportional between p/2 and CP – the semi principal diameter. (Or, alternatively that p is the third proportional to CP and DP.) Multiplying top and bottom by 2 this becomes:

2·CP:2·DP :: 2·DP:2·PN' = PP':DD' :: DD':p.

Draw the lines DC and CD'. Now,

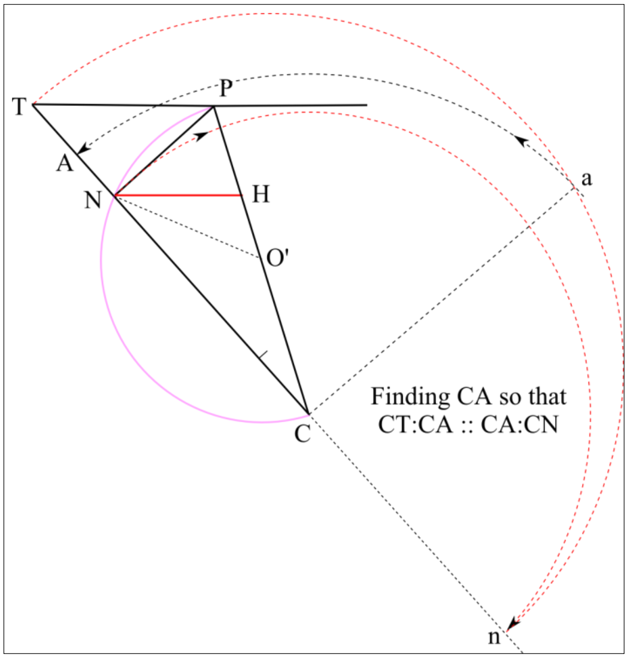
arc DT = arc TD' = arc DD'/2 and thus ∠DCT = ∠TCD' = ∠DCD'/2.

Hence, the point T lies on the bisector of ∠DCD' at the intersection of the bisector and PT. If the conjugate diameter is known point T can be located and CT drawn without needing to draw the circle. Point N can then be located by construction of the semi circle on CP or dropping a perpendicular to CT from P.

It should be noted that if the second diameter rather than p is known that CD rather than CN' will have to be bisected to locate i. It will be the same point as if CN' were used and consequently the circle will be the same.

One axis will lie along CT and the other will lie along CT'. CT' will be perpendicular to CT because the angle TCT' will be inscribed in a semi-circle. The major axis length will be equal DC + CD' and the minor axis length DC – CD'. This proof is left to the reader.

Notes and References



Constructing a Mean Proportional

The graphical method of constructing CA that I used without comment in the discussion of I.58 is shown here.

From C swing an arc of length NC intersecting the extension of TC at n. Construct a semi circle on Tn. Erect a perpendicular to Tn through C and intersecting the semi circle at a.

Swing an arc of length Ca from C intersecting CT at A. CA is the required mean proportional to CT and CN.

References

McKinney, Colin Bryan Powell. "*Conjugate diameters: Apollonius of Perga and Eutocius of Ascalon.*" dissertation, University of Iowa, 2010. http://ir.uiowa.edu/etd/711/

Heath, T.L. "*Apollonius of Perga: Treatise on Conic Sections*." Available from several online sources including: http://ebooks.adelaide.edu.au/a/apollonius_of_perga/conic/

In a footnote to Exercise 299, page 125, of "*The Ancient and Modern Geometry of Conics"*, by Charles Taylor references Tucker, R. "Construction for Axes of Ellipse." in the "*Oxford, Cambridge, and Dublin Messenger of Mathematics*", vol. III, pp. 151 (1866). The figures are missing from the scanned version of the latter publication that is available on Google Books. Ty Harness of Ty Harness Co. came to my rescue and produced working drawings. The idea for my method of finding N using the conjugate diameters came from these articles and Ty's drawings in particular.

The method of finding the axis using a circle on the tangent to the ellipse at the end of the principal axis came from Thomas Bradley's, "*Practical Geometry, Linear Perspective, and Projection,*" 1934 where it is offered without proof. Any errors in the proof are my own.